

POLES OF MAXIMAL ORDER OF MOTIVIC ZETA FUNCTIONS

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ABSTRACT. We prove a 1999 conjecture of Veys, which says that the opposite of the log canonical threshold is the only possible pole of maximal order of Denef and Loeser's motivic zeta function associated with a germ of a regular function on a smooth variety over a field of characteristic zero. We apply similar methods to study the weight function on the Berkovich skeleton associated with a degeneration of Calabi-Yau varieties. Our results suggest that the weight function induces a flow on the non-archimedean analytification of the degeneration towards the Kontsevich-Soibelman skeleton.

1. INTRODUCTION

(1.1) Let k be a field of characteristic zero and set $R = k[[t]]$ and $K = k((t))$. We endow K with its t -adic absolute value $|x| = \exp(-\text{ord}_t x)$. Let X be a connected smooth k -variety and let

$$f : X \rightarrow \text{Spec } k[t]$$

be a non-constant regular function on X . We set $\mathcal{X} = X \times_{k[t]} R$ and we denote by $\widehat{\mathcal{X}}_\eta$ the generic fiber of the t -adic completion $\widehat{\mathcal{X}}$ of \mathcal{X} ; this is a smooth K -analytic space. In [MN13], Mustaș and the first-named author defined the *weight function*

$$\text{wt}_{(f)} : \mathcal{X}_\eta \rightarrow \mathbb{R} \cup \{+\infty\}$$

that measures the singularities of the zero locus of f . It is closely related to the thinness function of [BFJ08] and the log discrepancy function of [JM11]. If v is the divisorial point of $\widehat{\mathcal{X}}_\eta$ associated with a prime divisor E on a birational modification of X , then $\text{wt}_{(f)}(v) = \nu_E/N_E$ where ν_E is the log discrepancy of E with respect to the pair $(X, (f))$ and N_E is the vanishing order of f along E . The minimal value of $\text{wt}_{(f)}$ on \mathcal{X}_η is precisely the log canonical threshold of f . Every log resolution $h : X' \rightarrow X$ of f gives rise to a Berkovich skeleton in $\widehat{\mathcal{X}}_\eta$ that is canonically homeomorphic to the dual complex of the strict normal crossings divisor $(f \circ h)$ on X' . The weight function $\text{wt}_{(f)}$ is affine on each face of this skeleton. We will apply techniques from the Minimal Model Program (MMP) to prove that, if $\text{wt}_{(f)}$ is constant on a maximal face of the Berkovich skeleton, then its value is equal to the log canonical threshold of f . To be precise, this property holds only locally over X ; we refer to Theorem 2.4 for the exact statement.

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(1.2) This result has interesting consequences for the so-called *motivic zeta function* $Z_{f,x}(s)$ of f at a closed point x of X . This is a rich invariant of the singularity of f at x that was defined by Denef and Loeser using motivic integration (see [DL01] for a nice introduction). The motivic zeta function is a rational function over a suitable coefficient ring, and it is a longstanding problem to understand the nature of its poles (or the poles of closely related invariants, such as the topological zeta function or Igusa's p -adic zeta function). The Monodromy Conjecture predicts that every pole of the motivic zeta function is a root of the Bernstein polynomial of f . The function $Z_{f,x}(s)$ has an explicit expression in terms of the geometry of the log resolution h , and this expression implies that the order of a pole is at most $n = \dim(X)$. Veys conjectured in [LV99] that, if the topological zeta function has a pole of order n , then this pole is the largest pole of the topological zeta function. We will deduce from Theorem 2.4 the following stronger form of Veys's conjecture. We denote by $\text{lct}_x(f)$ the log canonical threshold of f at x .

Theorem 3.5 (Veys's Conjecture). *The motivic zeta function $Z_{f,x}(s)$ has a pole at $s = -\text{lct}_x(f)$, and this is its largest pole. Conversely, if s_0 is a pole of order n of $Z_{f,x}(s)$, then $s_0 = -\text{lct}_x(f)$. In particular, s_0 is a root of the Bernstein polynomial of f .*

This statement implies the original conjecture of Veys because the order of s as a pole of the motivic zeta function is at least the order of s as a pole of the topological zeta function, since the latter is a specialization of the former.

(1.3) Theorem 2.4 has a natural counterpart for degenerations of Calabi-Yau varieties. Let X be a geometrically connected smooth projective K -variety with trivial canonical sheaf, and let ω be a volume form on X . We denote by X^{an} the Berkovich analytification of X . In [MN13], Mustařă and the first-named author also defined the weight function

$$\text{wt}_\omega : X^{\text{an}} \rightarrow \mathbb{R} \cup \{+\infty\}$$

that measures the degeneration of X at $t = 0$. The locus where wt_ω reaches its minimal value is independent of ω . It is called the *essential skeleton* of X and denoted by $\text{Sk}(X)$. The essential skeleton is a non-empty compact subspace of X^{an} with a canonical piecewise integral affine structure. This object was introduced by Kontsevich and Soibelman in their non-archimedean interpretation of Mirror Symmetry [KS06].

The essential skeleton can be computed as follows. Let \mathcal{X} be a regular proper R -model of X whose special fiber \mathcal{X}_k is a divisor with strict normal crossings. Then there exists a canonical embedding of the dual complex of \mathcal{X}_k in X^{an} . The image of this embedding is called the Berkovich skeleton of \mathcal{X} and denoted by $\text{Sk}(\mathcal{X})$. It follows from techniques introduced by Berkovich [Be99] and Thuillier [Th07] that $\text{Sk}(\mathcal{X})$ is a strong deformation retract of X^{an} . The weight function wt_ω can reach its minimal value only at points of $\text{Sk}(\mathcal{X})$, and it is affine on every face of $\text{Sk}(\mathcal{X})$. It follows that the essential skeleton $\text{Sk}(X)$ is a union of faces of $\text{Sk}(\mathcal{X})$ (see [MN13, 4.5.5]). We will prove the following analog of Theorem 2.4.

Theorem 4.6. *If τ is a maximal face of $\mathrm{Sk}(\mathcal{X})$ and wt_ω is constant on τ with value w , then w must be equal to the minimal value of wt_ω on X^{an} . Thus τ is contained in the essential skeleton $\mathrm{Sk}(X)$.*

(1.4) In [NX13] we proved that $\mathrm{Sk}(X)$ is equal to the Berkovich skeleton of any good minimal *dlt*-model of X over R (the kind of model produced by the MMP). We then deduced from the results in [dFKX12], obtained by a detailed analysis of the steps in the MMP, that the essential skeleton $\mathrm{Sk}(X)$ is a strong deformation retract of X^{an} . It seems plausible that one can use the weight function to create a natural flow on X^{an} in the direction of decreasing values of wt_ω , and use this flow to contract X^{an} onto the subspace $\mathrm{Sk}(X)$ where wt_ω takes its minimal value. Theorem 4.6 supports this strategy; further evidence is provided by the following result.

Theorem 4.8. *For every real number w we denote by $\mathrm{Sk}(\mathcal{X})^{\leq w}$ the subcomplex of $\mathrm{Sk}(\mathcal{X})$ spanned by the vertices where the value of wt_ω is at most w . Then there exists a collapse of $\mathrm{Sk}(\mathcal{X})$ to the essential skeleton $\mathrm{Sk}(X)$ which simultaneously collapses $\mathrm{Sk}(\mathcal{X})^{\leq w}$ to $\mathrm{Sk}(X)$ for all w greater than the minimal value of wt_ω on X^{an} . In particular, $\mathrm{Sk}(X)$ is a strong deformation retract of $\mathrm{Sk}(\mathcal{X})^{\leq w}$.*

(1.5) The paper is organized as follows. In Section 2 we deduce from the MMP the main technical result needed to prove Veys’s conjecture (Theorem 2.4). The proof of the conjecture is given in Section 3 (Theorem 3.5). To keep the proof as accessible as possible we avoided the language of weight functions on Berkovich spaces in these sections, although this interpretation was an important guide to obtain the results. In Section 4 we explain the relation with weight functions and we prove the analogous result for degenerations of Calabi-Yau varieties (Theorem 4.6), together with our result on the level sets of the weight function (Theorem 4.8).

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2. MAXIMAL INTERSECTIONS WITH EQUAL WEIGHTS

(2.1) We fix a base field k of characteristic zero. Let X be a connected smooth k -variety of dimension n and let Δ be an effective \mathbb{Q} -divisor on X . Let v be a divisorial valuation on X with center contained in Δ . This means that v is a real valuation on the function field $k(X)$ and there exist a birational morphism $h : Y \rightarrow X$ of k -varieties, with Y normal, and a prime component E of $h^*\Delta$ such that v is a real multiple of the valuation ord_E associated with E . We denote by N_E the multiplicity of $h^*\Delta$ along E and

by $\nu_E - 1$ the multiplicity of the relative canonical divisor $K_{Y/X}$ along E . We set

$$\mathrm{wt}_\Delta(v) = \frac{\nu_E}{N_E}$$

and we call this positive rational number the weight of Δ at v . This definition only depends on v , and not on the choice of the model Y . Note that $\mathrm{wt}_\Delta(v) = \mathrm{wt}_\Delta(\mathrm{ord}_E)$. For this reason, we will often denote $\mathrm{wt}_\Delta(v)$ by $\mathrm{wt}_\Delta(E)$.

(2.2) We fix a point x on X . The log canonical threshold of (X, Δ) at x is defined as

$$\mathrm{lct}_x(X, \Delta) = \inf_v \{\mathrm{wt}_\Delta(v)\},$$

where v runs through the set of divisorial valuations on X whose center lies in Δ and contains x . It is well known that, in order to compute this infimum, it suffices to let v run through the set of divisorial valuations associated with the prime components of the total transform of Δ on some log resolution of (X, Δ) .

(2.3) Let $h : Y \rightarrow X$ be a log resolution of (X, Δ) . We write

$$h^*\Delta = \sum_{i \in I} N_i E_i, \quad K_{Y/X} = \sum_{i \in I} (\nu_i - 1) E_i.$$

For every non-empty subset I' of I , we set $E_{I'} = \cap_{i \in I'} E_i$. Let J be a non-empty subset of I and let C be a connected component of E_J . We assume that the intersection $h^{-1}(x) \cap C$ is non-empty but $h^{-1}(x) \cap C \cap E_i$ is empty for every i in $I \setminus J$. Our main technical result is the following.

Theorem 2.4. *We keep the notations and assumptions of (2.3). If we assume that the value $\mathrm{wt}_\Delta(E_j)$ is the same for all j in J and we denote this value by w , then we have $w = \mathrm{lct}_x(X, \Delta)$.*

Proof. We write Δ as a sum $A + B$ of effective divisors without common components such that $wA \leq A_{\mathrm{red}}$ and either $wB > B_{\mathrm{red}}$ or $B = 0$. We define a new divisor Δ' on X by

$$\Delta' = wA + B_{\mathrm{red}} = \min\{w\Delta, \Delta_{\mathrm{red}}\}$$

where the minimum is taken componentwise. We set $Y_0 = Y$ and $\Delta_0 = h_*^{-1}(\Delta') + (K_{Y/X})_{\mathrm{red}}$. Then we run the relative MMP for the pair (Y_0, Δ_0) over X with scaling of some ample divisor. Since $\mathrm{Supp}(\Delta_0) = \mathrm{Supp}(h^*\Delta)$, we know that for sufficiently small $\varepsilon > 0$, this is the same as running the relative MMP for the *klt* pair $(Y_0, \Delta_0 - \varepsilon h^*\Delta)$. Hence, it follows from [BCHM10] that this MMP terminates with a minimal model.

The outcome is a series of birational maps

$$Y = Y_0 \dashrightarrow Y_1 \dashrightarrow \dots \dashrightarrow Y_m$$

where each of the Y_i is a \mathbb{Q} -factorial normal projective X -scheme and each of the birational maps is a map of X -schemes. If we denote by Δ_i the pushforward of the divisor Δ_0 to Y_i , then the pair (Y_i, Δ_i) is *dlt* for every i , and $K_{Y_m} + \Delta_m$ is nef over X .

Claim. *If we denote by ξ the generic point of C , then the birational map $Y_0 \dashrightarrow Y_m$ is an open embedding on some open neighbourhood of ξ in Y_0 .*

Proof. Let ℓ be an element of $\{-1, 0, \dots, m-1\}$. We will prove by induction on ℓ that the map $Y_0 \dashrightarrow Y_{\ell+1}$ is defined at ξ and that it is an open embedding on some open neighbourhood of ξ in Y_0 . This is trivial for $\ell = -1$, so that we may assume that $\ell \geq 0$ and that the property holds for $Y_0 \dashrightarrow Y_\ell$. With a slight abuse of notation, we will again write E_j for the pushforward of E_j to Y_ℓ , for every j in J . We write $\Delta_\ell^{\leq 1}$ for the reduced divisor on Y_ℓ consisting of the components of multiplicity one in Δ_ℓ .

Let y be a point of C lying over $x \in X$. The birational map $Y_\ell \dashrightarrow Y_{\ell+1}$ is either a divisorial contraction or a flip. In both cases, it is induced by an extremal ray R of $\overline{NE}(Y_\ell/X)$ such that

$$(2.5) \quad R \cdot (K_{Y_\ell} + \Delta_\ell) < 0.$$

We denote by $g : Y_\ell \rightarrow Z$ the contraction of R . Since the pair (Y_ℓ, Δ_ℓ) is *dlt*, its log canonical centers are precisely the connected components of subsets of the form $D_1 \cap \dots \cap D_r$ where D_1, \dots, D_r are prime components of $\Delta_\ell^{\leq 1}$. A special case of [Am03, 6.6] (see also [dFKX12, Prop. 25]) tells us that the set S of log canonical centers of (Y_ℓ, Δ_ℓ) intersecting the fiber $g^{-1}(g(y))$ has a unique minimal element. But C is such a minimal element, because we assumed that the intersection $h^{-1}(x) \cap C \cap E_i$ is empty for every i in $I \setminus J$.

Now suppose that g contracts a curve passing through y ; the class of any such curve generates the ray R . Then $E \cdot R = 0$ for every prime component E of $\Delta_\ell^{\leq 1}$ that is not one of the components E_j with $j \in J$. Otherwise, E would meet $g^{-1}(g(y))$ and S would have a minimal element contained in E , which is impossible since E cannot meet $C \times_X x$ by our assumptions on C .

In particular, $E \cdot R = 0$ for every component of $(\Delta_\ell)_{\text{red}} - \sum_{j \in J} E_j$ that is contracted on X or contained in the strict transform of B . Denoting by f the morphism $f : Y_\ell \rightarrow X$, we compute:

$$\begin{aligned} R \cdot (K_{Y_\ell} + \Delta_\ell) &= R \cdot (K_{Y_\ell} + \Delta_\ell - f^*(K_X + w\Delta)) \\ &= R \cdot (K_{Y_\ell/X} + (K_{Y_\ell/X})_{\text{red}} + f_*^{-1}\Delta' - f^*(w\Delta)) \\ &= R \cdot \left(\sum_{j \in J} (\nu_j - wN_j)E_j + f_*^{-1}(B_{\text{red}} - wB) \right) \\ &= 0. \end{aligned}$$

This contradicts the inequality (2.5). We conclude that g cannot contract a curve through y . Therefore, $Y_\ell \dashrightarrow Y_{\ell+1}$ must be an open embedding on some open neighbourhood of y in Y_ℓ . \square

Using this result, we can finish the proof of Theorem 2.4. We denote by g the morphism $g : Y_m \rightarrow X$ and we write E_j for the image of E_j in Y_m , for every $j \in J$. Now consider the divisor

$$D = K_{Y_m} + \Delta_m - g^*(K_X + w\Delta)$$

on Y_m . This divisor is nef over X . We can write D as

$$D = D_{\text{exc}} - g_*^{-1}(wB - B_{\text{red}})$$

where the divisor D_{exc} is g -exceptional and $wB - B_{\text{red}}$ is effective. The negativity lemma [KM98, 3.39] implies that $-D$ is effective and that the

support of D is a union of fibers of g . But for every j in J , the multiplicity of D along E_j is equal to zero and thus $g^{-1}(x) \cap \text{Supp}(D) = \emptyset$. This means that locally around x , we have $B = 0$ and $\Delta' = w\Delta$. It also follows that

$$(Y_m, g^*(w\Delta) - K_{Y_m/X}) = (Y_m, \Delta_m)$$

over some open neighbourhood of x in X . This pair is *dlt* and Δ_m contains components of multiplicity one intersecting $g^{-1}(x)$ (for instance, the components E_j with $j \in J$). It follows that $w = \text{lct}_x(X, \Delta)$. \square

(2.6) Now suppose that we are still in the situation of Theorem 2.4, that Δ is an effective \mathbb{Z} -divisor on X , and that the cardinality of J is equal to n , the dimension of X . In [LV99, Thm. 3.2], Laeremans and Veys proved by combinatorial arguments that in this case, the weight w is of the form $1/N$ for some positive integer N . We will now explain how one can deduce this result from the MMP.

Theorem 2.4 implies that w equals the log canonical threshold $\text{lct}_x(X, \Delta)$. We refer to [dFKX12, Def. 13] for the definition of a *dlt*-modification $(Z^{\text{dlt}}, D^{\text{dlt}})$ of a pair (Z, D) . The log canonical centers of $(Z^{\text{dlt}}, D^{\text{dlt}})$ are the irreducible components of the intersections of prime components in $[D^{\text{dlt}}]$. If $(X^{\text{dlt}}, \Delta^{\text{dlt}})$ is a *dlt*-modification of (X, Δ) , then $(X^{\text{dlt}}, \Delta^{\text{dlt}})$ has a log canonical center of dimension zero (either apply [dFKX12, Prop. 11] or use [MN13, 6.4.1] to see that the dual complex of Δ^{dlt} has a face of dimension n). Thus [LV99, Thm. 3.2] is a consequence of the following result, applied to $(Z, D) = (X, \Delta)$ (after replacing X by a sufficiently small open neighbourhood of x).

Proposition 2.7. *Let Z be a smooth k -variety and let D_0 be an effective \mathbb{Z} -divisor on Z . Denote by c the log canonical threshold of the pair (Z, D_0) , and set $D = cD_0$. Let $h^{\text{dlt}} : (Z^{\text{dlt}}, D^{\text{dlt}}) \rightarrow (Z, D)$ be a *dlt*-modification of (Z, D) , where D^{dlt} denotes the log pullback of D to Z^{dlt} :*

$$(h^{\text{dlt}})^*(K_Z + D) = K_{Z^{\text{dlt}}} + D^{\text{dlt}}.$$

Assume that $(Z^{\text{dlt}}, D^{\text{dlt}})$ has a log canonical center of dimension zero. Then $c = 1/N$ for some positive integer N .

Proof. We can assume Z^{dlt} is \mathbb{Q} -factorial. We denote by E the h^{dlt} -exceptional part of D^{dlt} . We can assume that no component of the strict transform of D has multiplicity one in D^{dlt} , since otherwise the conclusion follows trivially.

We run a relative MMP for (Z^{dlt}, E) over Z with a scaling of an ample divisor, and since Z is smooth, we know it eventually contracts to Z , i.e., we will have a sequence

$$Z^{\text{dlt}} = Z_0 \dashrightarrow \cdots \dashrightarrow Z_m = Z.$$

Let Z_i be the first step such that $Z_i \dashrightarrow Z_{i+1}$ contracts a 0-dimensional lc center z . Applying [dFKX12, Lemma 23] to $Z_i \rightarrow Z$, we know that the assumption of [dFKX12, Thm. 19] holds. Therefore we conclude that $Z_i \dashrightarrow Z_{i+1}$ contracts a 1-dimensional log canonical center C containing z . Denoting by E_i the pushforward of E to Z_i , it follows from adjunction that

$$(K_{Z_i} + E_i)|_C = K_C + \text{Diff}_C(E_i)$$

for some effective \mathbb{Q} -divisor $\text{Diff}_C(E_i)$ (see [Ko13, 4.2]), which is anti-ample. In particular, C is a smooth rational curve.

Let D_i be the pushforward of D^{dlt} on Z_i . Applying adjunction (see [Ko13, 3.45 and 4.2]), we know that

$$(2.8) \quad (K_{Z_i} + D_i)|_C = K_C + \text{Diff}_C(D_i) \sim_{\mathbb{Q}} 0,$$

as it coincides with the pullback of $K_Z + D$ to C . Thus C meets the strict transform of D on Z_i , which is $D_i - E_i$.

By the definition of the different divisor $\text{Diff}_C(D_i)$, we know that its coefficients are contained in the set

$$\left\{ \frac{Nc + p - 1}{p} \mid p \in \mathbb{Z}_{>0} \text{ and } N \in \mathbb{Z}_{\geq 0} \right\},$$

where $N > 0$ if and only if the point is contained in the strict transform of D . Since $\text{Diff}_C(D_i) - z \geq 0$, we deduce from (2.8) by taking degrees an equation of the form

$$\sum_j \frac{N_j c + p_j - 1}{p_j} = 1.$$

We immediately see that there is at most one point in $\text{Supp}(\text{Diff}_C(D_i) - z)$ whose coefficient is of the form $\frac{Nc + p - 1}{p}$ with $p \geq 2$. In other words, the equation can be rewritten as

$$N_1 c + \frac{N_2 c + p - 1}{p} = 1,$$

which implies $c = \frac{1}{pN_1 + N_2}$ with $(N_1, N_2) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \setminus \{(0, 0)\}$. \square

3. POLES OF MOTIVIC ZETA FUNCTIONS

(3.1) Let X be a connected smooth k -variety, let x be a closed point on X , and let f be a regular function on X such that $f(x) = 0$. Denef and Loeser defined the *motivic zeta function* $Z_{f,x}(s)$ of the germ of f at x , an invariant that measures the singularity of f at the point x . It is a power series in \mathbb{L}^{-s} over a certain Grothendieck ring $\mathcal{M}_x^{\hat{\mu}}$ of $\kappa(x)$ -varieties with an action of the profinite group scheme $\hat{\mu}$ of roots of unity over k . Here $\kappa(x)$ denotes the residue field of X at x and \mathbb{L}^{-s} should be viewed as a formal variable. The zeta function $Z_{f,x}(s)$ is obtained from the generating series $Z_f(T)$ defined in [DL01, §3.2] by applying the base change morphism $\mathcal{M}_{X_0}^{\hat{\mu}} \rightarrow \mathcal{M}_x^{\hat{\mu}}$ to its coefficients and setting $T = \mathbb{L}^{-s}$. Closely related invariants are the so-called *naïve* motivic zeta function $Z_{f,x}^{\text{naïve}}(s)$, which is a power series in \mathbb{L}^{-s} over the Grothendieck ring \mathcal{M}_x of $\kappa(x)$ -varieties without group action, and the topological zeta function $Z_{f,x}^{\text{top}}(s)$, which is an element of the field of rational functions $\mathbb{Q}(s)$.

(3.2) What is important for our purposes is that each of these zeta functions can be explicitly computed on a log-resolution $h : Y \rightarrow X$ of (X, Δ) , where we set $\Delta = (f)$. We write $h^* \Delta = \sum_{i \in I} N_i E_i$ and $K_{Y/X} = \sum_{i \in I} (\nu_i - 1) E_i$. For every subset J of I , we set $E_J = \cap_{j \in J} E_j$

and $E_J^o = E_J \setminus (\cup_{i \notin J} E_i)$. Then we have the following expressions for the zeta functions introduced above (see [DL01, §3.3 and §3.4]):

$$\begin{aligned} Z_{f,x}(s) &= \sum_{J \subset I} (\mathbb{L} - 1)^{|J|-1} [\tilde{E}_J^o \times_X x] \prod_{j \in J} \frac{\mathbb{L}^{-\nu_j - N_j s}}{1 - \mathbb{L}^{-\nu_j - N_j s}}, \\ Z_{f,x}^{\text{naive}}(s) &= \sum_{J \subset I} (\mathbb{L} - 1)^{|J|} [E_J^o \times_X x] \prod_{j \in J} \frac{\mathbb{L}^{-\nu_j - N_j s}}{1 - \mathbb{L}^{-\nu_j - N_j s}}, \\ Z_{f,x}^{\text{top}}(s) &= \sum_{J \subset I} \chi(E_J^o \times_X x) \prod_{j \in J} \frac{1}{N_j s + \nu_j}. \end{aligned}$$

Here \mathbb{L} denotes the class of the affine line \mathbb{A}_k^1 , \tilde{E}_J^o is a certain finite étale cover of E_J^o with an action of the group scheme $\hat{\mu}$, and $\chi(\cdot)$ denotes the ℓ -adic Euler characteristic (which coincides with the singular Euler characteristic for the complex topology if k is a subfield of \mathbb{C}).

(3.3) It is obvious from these explicit formulas that each pole is of the form $\text{wt}_\Delta(E_i) = -\nu_i/N_i$ for some $i \in I$ (see Remark 3.7 for a precise definition of the poles). Thus the largest possible pole is the negative of the log canonical threshold

$$\text{lct}_x(X, \Delta) = \min \left\{ \frac{\nu_i}{N_i} \mid i \in I, x \in h(E_i) \right\}$$

of f at x . However, in practice most of these candidate-poles will not be actual poles due to cancellations in the formulas. This phenomenon would be explained by Denef and Loeser's motivic monodromy conjecture, which predicts that every pole of each of these three zeta functions is a root of the Bernstein polynomial of f . This conjecture was motivated by an analogous conjecture of Igusa for p -adic local zeta functions of polynomials over number fields. Recall that $-\text{lct}_x(X, \Delta)$ is always the largest root of the Bernstein polynomial of f at x ; see for instance [Ko97, 10.6]. The monodromy conjecture has been proven if $\dim(X) = 2$ [Lo88, Ro04] and also for some special classes of singularities, but it remains wide open in general. We refer to [Ni10] for a gentle introduction and a survey of some known results.

(3.4) It is also clear from the formulas in (3.2) that the order of a pole is at most $n = \dim(X)$, since E_J is empty for every subset J of I of cardinality strictly larger than n . In [LV99, 0.2], Veys made the following conjecture.

Conjecture (Veys). *If $Z_{f,x}^{\text{top}}(s)$ has a pole s_0 of order n , then s_0 must be the largest pole of $Z_{f,x}^{\text{top}}(s)$.*

Veys proved this statement if $n = 2$ [Ve95, 4.2] and also if f is a polynomial that is non-degenerate with respect to its Newton polyhedron [LV99, 2.4], but these were the only cases known so far. We can deduce from Theorem 2.4 the following refinement of Veys's conjecture.

Theorem 3.5. *Let X be a connected smooth k -variety of dimension n , let x be a k -rational point on X , and let f be a non-constant regular function on X . Let $h : Y \rightarrow X$ be a log resolution for f as in (3.2), and denote by m the*

largest positive integer such that there exists a subset J of I of cardinality m with $E_J \cap h^{-1}(x) \neq \emptyset$ and $\nu_j/N_j = \text{lct}_x(X, \Delta)$ for every $j \in J$. Then the following properties hold.

- (1) The motivic zeta functions $Z_{f,x}(s)$ and $Z_{f,x}^{\text{naive}}(s)$ have a pole of order m at $s = -\text{lct}_x(X, \Delta)$, and this is their largest pole. If $m = n$ then the topological zeta function $Z_{f,x}^{\text{top}}(s)$ has a pole of order n at $s = -\text{lct}_x(X, \Delta)$, and this is its largest pole.
- (2) Conversely, if s_0 is a pole of order n of $Z_{f,x}(s)$, $Z_{f,x}^{\text{naive}}(s)$ or $Z_{f,x}^{\text{top}}(s)$, then $s_0 = -\text{lct}_x(X, \Delta)$ and $m = n$. Moreover, s_0 is of the form $-1/N$ for some positive integer N .

Proof. (1) This result is more or less folklore, and it can be proven by straightforward computation. Note that it is clear from the expressions in (3.2) that $Z_{f,x}(s)$, $Z_{f,x}^{\text{naive}}(s)$ and $Z_{f,x}^{\text{top}}(s)$ have no poles that are strictly larger than $-\text{lct}_x(X, \Delta)$, and that the order of $-\text{lct}_x(X, \Delta)$ as a pole is at most m . Now we specialize $Z_{f,x}(s)$ and $Z_{f,x}^{\text{naive}}(s)$ to elements in $\mathbb{Z}[u, u^{-1}][\mathbb{L}^{-s}]$ by means of the ring morphisms

$$\mathcal{M}_x^{\hat{\mu}} \rightarrow \mathcal{M}_x \rightarrow \mathbb{Z}[u, u^{-1}]$$

where the first morphism simply forgets the $\hat{\mu}$ -action and the second one sends the class of a $\kappa(x)$ -variety Z to the Poincaré polynomial $P_Z(u)$ of Z (see [Ni11, §8]). What matters here is that $P_Z(u)$ is a non-zero polynomial with positive leading coefficient if Z is non-empty. Using this property, one easily verifies that the residue at the expected pole of order m at $s = -\text{lct}_x(X, \Delta)$ is different from zero. Likewise, if $m = n$ then one immediately sees that the residue of the expected pole of $Z_{f,x}^{\text{top}}(s)$ of order n at $s = -\text{lct}_x(X, \Delta)$ is positive.

(2) If s_0 is a pole of order n then it follows from the explicit formulas for the zeta functions in (3.2) that there must exist a subset J of I of cardinality n such that $E_J \cap h^{-1}(x)$ is non-empty and $s_0 = -\nu_j/N_j$ for every j in J . By Theorem 2.4, this can only happen when $s_0 = -\text{lct}_x(X, \Delta)$ and $m = n$. As we mentioned in 2.6, it was already shown in [LV99] that s_0 is of the form $-1/N$; we gave another proof of this property in Proposition 2.7. \square

(3.6) In particular, a pole of order n of $Z_{f,x}(s)$, $Z_{f,x}^{\text{naive}}(s)$ or $Z_{f,x}^{\text{top}}(s)$ is always a root of the Bernstein polynomial of f , as predicted by the monodromy conjecture. If f has an isolated singularity at x then it is even a root of order n , by the proof of Theorem 1 in [MTV09]. Beware that if $m < n$, we do not *not* claim that the value $-\text{lct}_x(X, \Delta)$ is a pole of order m of the topological zeta function $Z_{f,x}^{\text{top}}(s)$. The Euler characteristic is too crude as an invariant to guarantee that the residue at the expected pole is non-zero. The proof of Theorem 3.5(2) is also valid for Igusa's local zeta function of a polynomial f over a p -adic field K , as can be seen from Igusa's computation of the zeta function on a log resolution of (X, Δ) in [Ig00, 8.2.1].

Remark 3.7. Since the Grothendieck ring $\mathcal{M}_x^{\hat{\mu}}$ is not a domain, one should specify what is meant by a pole of a rational function over $\mathcal{M}_x^{\hat{\mu}}$. The

definition we use in 3.5 is the following: if $Z(\mathbb{L}^{-s})$ is an element of

$$\mathcal{M}_x^{\hat{\mu}} \left[\mathbb{L}^{-s}, \frac{1}{1 - \mathbb{L}^{a-bs}} \right]_{(a,b) \in \mathbb{Z} \times \mathbb{Z}_{>0}} \subset \mathcal{M}_x^{\hat{\mu}}[[\mathbb{L}^{-s}]],$$

s_0 is a rational number and m is a non-negative integer, then we say that $Z(\mathbb{L}^{-s})$ has a pole at s_0 of order at most m if we find a set \mathcal{S} consisting of multisets in $\mathbb{Z} \times \mathbb{Z}_{>0}$ such that each element of \mathcal{S} contains at most m elements (a, b) such that $a/b = s_0$ and $Z(\mathbb{L}^{-s})$ belongs to the sub- $\mathcal{M}_x^{\hat{\mu}}[[\mathbb{L}^{-s}]]$ -module of $\mathcal{M}_x^{\hat{\mu}}[[\mathbb{L}^{-s}]]$ generated by

$$\left\{ \frac{1}{\prod_{(a,b) \in S} (1 - \mathbb{L}^{a-bs})} \mid S \in \mathcal{S} \right\}.$$

The same remark applies to \mathcal{M}_x .

4. THE WEIGHT FUNCTION AND THE ESSENTIAL SKELETON

(4.1) Theorem 2.4 can be rephrased in terms of skeleta in Berkovich spaces. We will briefly explain this reformulation and then prove the natural counterpart of Theorem 2.4 for Kontsevich-Soibelman skeleta of degenerations of projective varieties. Let X be a connected smooth k -variety of dimension n , let $f : X \rightarrow \text{Spec } k[t]$ be a regular function on X and let x be a closed point in the divisor $\Delta = (f)$. We set $R = k[[t]]$ and $K = k((t))$ and we endow R with its t -adic topology and K with its t -adic absolute value $|x| = \exp(-\text{ord}_t x)$. We set $\mathcal{X} = X \times_{k[[t]]} R$ and we denote by $\widehat{\mathcal{X}}$ the formal t -adic completion of \mathcal{X} . We write $\widehat{\mathcal{X}}_\eta$ for the generic fiber of $\widehat{\mathcal{X}}$ and $\widehat{\mathcal{X}}_k = \widehat{\mathcal{X}} \times_R k$ for its special fiber. Then $\widehat{\mathcal{X}}$ is a separated formal scheme of finite type over R , and $\widehat{\mathcal{X}}_\eta$ is a smooth K -analytic space. We denote by $\text{sp}_{\mathcal{X}} : \widehat{\mathcal{X}}_\eta \rightarrow \widehat{\mathcal{X}}_k$ the specialization map. We define the weight function

$$\text{wt}_\Delta : \widehat{\mathcal{X}}_\eta \rightarrow \mathbb{R} \cup \{+\infty\}$$

as the restriction to $\widehat{\mathcal{X}}_\eta$ of the weight function $\text{wt}_{\mathcal{I}}$ from [MN13, §6.1], with $\mathcal{I} = (f)$ (in the notation of [MN13, §6.1], $\widehat{\mathcal{X}}_\eta$ is the subspace of \widehat{X}_η defined by the equation $|f| = 1/e$; see [MN13, 6.3.4]). This weight function is closely related to the thinness function of [BFJ08] and the log discrepancy function of [JM11].

(4.2) Let $h : Y \rightarrow X$ be a log resolution of the pair (X, Δ) . The dual complex of the strict normal crossings divisor $h^*\Delta$ can be embedded in a natural way in the K -analytic space $\widehat{\mathcal{X}}_\eta$. The image of this embedding is the so-called Berkovich skeleton $\text{Sk}(\mathcal{Y})$ of $\mathcal{Y} = Y \times_{k[[t]]} R$; see for instance [MN13, §3.1]. Each prime component E of $h^*\Delta$ corresponds to a vertex of $\text{Sk}(\mathcal{Y})$, and the value of the weight function wt_Δ at this vertex is precisely the weight $\text{wt}_\Delta(E)$ defined in (2.1). Moreover, the weight function wt_Δ is affine on every face of $\text{Sk}(\mathcal{Y})$. Thus we can restate Theorem 2.4 as follows: if τ is a maximal element of the set of faces of $\text{Sk}(\mathcal{Y})$ that intersect $\text{sp}_{\mathcal{X}}^{-1}(x)$ and wt_Δ is constant on τ with value w , then $w = \text{lct}_x(X, \Delta)$.

(4.3) We will generalize this result to the following set-up. Let X be a geometrically connected smooth projective K -scheme with trivial canonical sheaf, and let ω be a volume form on X . Then on the K -analytic space X^{an} we can again consider a weight function

$$\text{wt}_\omega : X^{\text{an}} \rightarrow \mathbb{R} \cup \{+\infty\},$$

associated with the form ω . This function was defined in [MN13, 4.4.4]. It is bounded below and the set of points in X^{an} where it reaches its minimal value is a non-empty compact subspace of X^{an} that we call the *essential skeleton* of X and that we denote by $\text{Sk}(X)$; see [MN13, §4.6]. This definition does not depend on the choice of ω because multiplying ω with an element $a \in K^\times$ shifts the weight function by the t -adic valuation of a . The essential skeleton $\text{Sk}(X)$ was first considered by Kontsevich and Soibelman in their non-archimedean interpretation of Mirror Symmetry [KS06].

(4.4) Let \mathcal{X} be an *snc*-model of X over R , that is, a regular flat proper R -scheme endowed with an isomorphism of K -schemes $\mathcal{X}_K \rightarrow X$ such that the special fiber \mathcal{X}_k is a strict normal crossings divisor. Then \mathcal{X} again gives rise to a Berkovich skeleton $\text{Sk}(\mathcal{X})$ in X^{an} that is canonically homeomorphic to the dual complex of \mathcal{X}_k (see [MN13, §3.1]). The weight function wt_ω is affine on every face of $\text{Sk}(\mathcal{X})$, by [MN13, 4.3.3]. Moreover, $\text{Sk}(X)$ is contained in $\text{Sk}(\mathcal{X})$ by [MN13, 4.4.5(3)]. Thus $\text{Sk}(X)$ is a union of faces of $\text{Sk}(\mathcal{X})$ (see [MN13, 4.5.5] for a generalization of this result). If x is a vertex of $\text{Sk}(\mathcal{X})$ corresponding to a prime component E of \mathcal{X}_k , then

$$\text{wt}_\omega(x) = \text{wt}_\omega(E) := \frac{\nu}{N}.$$

Here N is the multiplicity of E in \mathcal{X}_k and $\nu - 1$ is the multiplicity of E in $\text{div}_{\mathcal{X}}(\omega)$, the divisor on \mathcal{X} associated with the rational section ω of the relative canonical line bundle $\omega_{\mathcal{X}/R}$. The following lemma reduces the study of the weight function on $\text{Sk}(\mathcal{X})$ to the case where \mathcal{X} is defined over an algebraic curve. We will need this reduction below to apply certain tools from the MMP.

Lemma 4.5. *Let X be a geometrically connected smooth projective K -scheme with trivial canonical sheaf, and let ω be a volume form on X . Let \mathcal{X} be an *snc*-model of X over R . Then we can always find the following objects.*

- (1) *A smooth curve \mathcal{C} over k , a k -rational point s on \mathcal{C} and a local parameter t on \mathcal{C} at s , which gives rise to a k -morphism $\text{Spec } R \rightarrow \mathcal{C}$. We set $C = \mathcal{C} \setminus \{s\}$.*
- (2) *A projective morphism $\mathcal{Y} \rightarrow \mathcal{C}$ with geometrically connected fibers such that $\mathcal{Y} \times_{\mathcal{C}} C \rightarrow C$ is smooth with trivial relative canonical sheaf, \mathcal{Y} is regular and $\mathcal{Y}_s = \mathcal{Y} \times_{\mathcal{C}} s$ is a divisor with strict normal crossings.*
- (3) *A relative volume form ω' on $\mathcal{Y} \times_{\mathcal{C}} C$ over C ; with a slight abuse of notation, we will denote the base change of ω' to $\mathcal{Y} \times_{\mathcal{C}} \text{Spec}(K)$ with the same symbol.*
- (4) *An isomorphism of simplicial complexes*

$$\text{Sk}(\mathcal{Y} \times_{\mathcal{C}} \text{Spec}(R)) \rightarrow \text{Sk}(\mathcal{X})$$

that identifies the weight function $\mathrm{wt}_{\omega'}$ on $\mathrm{Sk}(\mathcal{Y} \times_{\mathcal{C}} \mathrm{Spec}(R))$ with the weight function wt_{ω} on $\mathrm{Sk}(\mathcal{X})$.

Proof. The proof is similar to that of [NX13, 4.2.4]. Let N be a positive integer. By a standard spreading out argument combined with Greenberg Approximation, we find objects as in (1) and (2) together with an isomorphism of R -schemes

$$\varphi : \mathcal{X} \times_R R/(t^N) \rightarrow \mathcal{Y} \times_{\mathcal{C}} \mathrm{Spec}(R/(t^N)).$$

In particular, φ induces an isomorphism of k -schemes $\mathcal{X}_k \rightarrow \mathcal{Y}_s$ that we can use to identify the dual complex $\mathrm{Sk}(\mathcal{X})$ of \mathcal{X}_k with the dual complex $\mathrm{Sk}(\mathcal{Y})$ of \mathcal{Y}_s . We denote by S^+ the spectrum of R with its standard log structure and by \mathcal{X}^+ the scheme \mathcal{X} endowed with the divisorial log structure associated with \mathcal{X}_k . Likewise, we denote by \mathcal{C}^+ the curve \mathcal{C} with the log structure induced by s and by \mathcal{Y}^+ the scheme \mathcal{Y} with the divisorial log structure associated with \mathcal{Y}_s . The R -module

$$M = H^0(\mathcal{X}, \omega_{\mathcal{X}^+/S^+})$$

is free of rank one by [IKN05, 7.1]. Multiplying ω with t^a for some integer a shifts the weight function wt_{ω} by the constant a , so that we can assume that ω extends to a generator of M . But [IKN05, 7.1] also tells us that the $\mathcal{O}_{\mathcal{C}}$ -module $f_*\omega_{\mathcal{Y}^+/\mathcal{C}^+}$ is locally free of rank one and that its base change to $R/(t^N)$ is canonically isomorphic to $M \otimes_R R/(t^N)$. Shrinking \mathcal{C} around s if necessary, we can lift the class of ω in $M \otimes_R R/(t^N)$ to an element ω' of $H^0(\mathcal{Y}, \omega_{\mathcal{Y}^+/\mathcal{C}^+})$ that is a relative volume form over C . If N is sufficiently large, then the divisors of ω and ω' , viewed as sections of the line bundles $\omega_{\mathcal{X}^+/S^+}$ and $\omega_{\mathcal{Y}^+/\mathcal{C}^+}$, respectively, coincide (note that both divisors are supported on $\mathcal{Y}_s \cong \mathcal{X}_k$). Then it follows from the logarithmic interpretation of the weight function in [NX13, 3.2.2] that the restriction of wt_{ω} to $\mathrm{Sk}(\mathcal{X})$ coincides with the restriction of $\mathrm{wt}_{\omega'}$ to $\mathrm{Sk}(\mathcal{Y})$. \square

Theorem 4.6. *Let X be a geometrically connected smooth projective K -scheme with trivial canonical sheaf, and let ω be a volume form on X . Let \mathcal{X} be an snc-model of X over R and let τ be a maximal face of $\mathrm{Sk}(\mathcal{X})$ such that the weight function wt_{ω} is constant on τ with value w . Then w is the minimal value of wt_{ω} on X_K^{an} and τ is contained in the essential skeleton $\mathrm{Sk}(X)$.*

Proof. By Lemma 4.5 we can assume that \mathcal{X} and ω are defined over an algebraic curve. More precisely, we may assume that $X = \mathcal{Y} \times_{\mathcal{C}} \mathrm{Spec}(K)$, $\mathcal{X} = \mathcal{Y} \times_{\mathcal{C}} \mathrm{Spec}(R)$ and $\omega = \omega'$, where \mathcal{C} , \mathcal{Y} and ω' are taken as in the statement of Lemma 4.5. We will use similar arguments as in the proof of Theorem 2.4. We write $\mathcal{Y}_s = \sum_{i \in I} N_i E_i$. The face τ corresponds to a connected component U of $E_J = \bigcap_{j \in J} E_j$ for some non-empty subset J of I . The volume form ω' is a rational section of the relative canonical sheaf $\omega_{\mathcal{Y}/\mathcal{C}}$ and thus defines a divisor

$$\mathrm{div}_{\mathcal{Y}}(\omega') = \sum_{i \in I} (\nu_i - 1) E_i$$

on \mathcal{Y} . Our assumption that wt_ω is constant on τ with value w is equivalent to the property that

$$\text{div}_{\mathcal{Y}}(\omega') + (\mathcal{Y}_s)_{\text{red}} = w\mathcal{Y}_s$$

on some open neighbourhood of U in \mathcal{Y} .

We set $\Delta = (\mathcal{Y}_s)_{\text{red}}$ and we run an MMP with scaling of an ample divisor for the pair (\mathcal{Y}, Δ) over \mathcal{C} . This is the same as running a relative MMP for $(\mathcal{Y}, \Delta - \varepsilon\mathcal{Y}_s)$ for a sufficiently small $\varepsilon > 0$ such that the latter pair is *klt*. By [HX13, §2], the outcome is a series of birational maps

$$\mathcal{Y} = \mathcal{Y}_0 \dashrightarrow \mathcal{Y}_1 \dashrightarrow \dots \dashrightarrow \mathcal{Y}_m$$

where each of the \mathcal{Y}_i is a \mathbb{Q} -factorial normal projective \mathcal{C} -scheme and each of the birational maps is a map of \mathcal{C} -schemes whose restriction over C is an isomorphism. If we set $\Delta_i = (\mathcal{Y}_i)_{s, \text{red}}$ then the pair $(\mathcal{Y}_i, \Delta_i)$ is *dlt*, for every i . Moreover, $K_{\mathcal{Y}_m/\mathcal{C}} + \Delta_m$ is nef over \mathcal{C} , which implies that

$$K_{\mathcal{Y}_m/\mathcal{C}} + \Delta_m \sim_{\mathcal{C}, \mathbb{Q}} 0.$$

The same arguments as in the proof of Theorem 2.4 show that $\mathcal{Y} \dashrightarrow \mathcal{Y}_m$ is an open immersion on some open neighbourhood of U in \mathcal{X} . Thus

$$\text{div}_{\mathcal{Y}_m}(\omega') + \Delta_m = w(\mathcal{Y}_m)_s.$$

It now follows from [NX13, 3.3.4] that the minimal weight of ω on X^{an} is equal to w . \square

(4.7) We still denote by X a geometrically connected smooth projective K -scheme with trivial canonical sheaf, and by ω a volume form on X . Let \mathcal{X} be an *snc*-model of X over R . Then $\text{Sk}(\mathcal{X})$ is a strong deformation retract of X^{an} , by [NX13, 3.1.4]. In [NX13, 4.2.4] we deduced from the results in [dFKX12] that the essential skeleton $\text{Sk}(X)$ is a strong deformation retract of $\text{Sk}(\mathcal{X})$, and thus of the K -analytic space X^{an} . It seems natural to expect that the weight function induces a flow on X^{an} in the direction of decreasing values of wt_ω that contracts X^{an} onto the subspace $\text{Sk}(X)$ where wt_ω takes its minimal value. Theorem 4.6 supports this expectation. Further evidence is provided by the following theorem. For the definition of an elementary collapse we refer to Definition 18 in [dFKX12]; it is a combinatorial operation on simplicial complexes which is, in particular, a strong deformation retract. A collapse is a composition of elementary collapses.

Theorem 4.8. *Let X be a geometrically connected smooth projective K -scheme with trivial canonical sheaf, and let ω be a volume form on X . Let \mathcal{X} be a projective *snc*-model of X over R . For every real number w we denote by $\text{Sk}(\mathcal{X})^{\leq w}$ the subcomplex of $\text{Sk}(\mathcal{X})$ generated by the vertices where the value of wt_ω is at most w . Then there exists a collapse of $\text{Sk}(\mathcal{X})$ to the essential skeleton $\text{Sk}(X)$ that simultaneously collapses $\text{Sk}(\mathcal{X})^{\leq w}$ to $\text{Sk}(X)$ for all w greater than the minimal value of wt_ω on X^{an} .*

Proof. We can again assume that $X = \mathcal{Y} \times_{\mathcal{C}} \text{Spec}(K)$, $\mathcal{X} = \mathcal{Y} \times_{\mathcal{C}} \text{Spec}(R)$ and $\omega = \omega'$, where \mathcal{C} , \mathcal{Y} and ω' are taken as in the statement of Lemma 4.5. Denote by w_0 the minimal value of wt_ω on X^{an} . If we run a relative MMP

of $(\mathcal{Y}, (\mathcal{Y}_s)_{\text{red}})$ over \mathcal{C} with scaling of an ample divisor, then we obtain a sequence of birational maps of \mathcal{C} -schemes

$$(4.9) \quad \mathcal{Y} = \mathcal{Y}_0 \dashrightarrow \mathcal{Y}_1 \dashrightarrow \dots \dashrightarrow \mathcal{Y}_m$$

such that \mathcal{Y}_m is a minimal *dlt*-model. The skeleton $\text{Sk}(\mathcal{Y}_m)$ is equal to the essential skeleton $\text{Sk}(X)$ by [NX13, 3.3.4], and the MMP process induces a collapse of $\text{Sk}(\mathcal{Y})$ to $\text{Sk}(X)$ by [dFKX12, Cor. 22] (see also [NX13, 3.2.8]). We will now show that it simultaneously collapses $\text{Sk}(\mathcal{Y})^{\leq w}$ to $\text{Sk}(X)$ for all $w \geq w_0$.

We fix $w \geq w_0$ and we write $(\mathcal{Y}_s)_{\text{red}}$ as a sum of reduced effective divisors $(\mathcal{Y}_s)_{\text{red}} = A + B$ such that $\text{wt}_\omega(E) \leq w$ for every prime component E of A and $\text{wt}_\omega(E) > w$ for every prime component E of B . We choose $\varepsilon > 0$ sufficiently small and we set $\Delta = A + (1 - \varepsilon)B$. We denote by Δ_i , A_i and B_i the pushforwards to \mathcal{Y}_i of Δ , A and B , respectively, for every i in $\{0, \dots, m\}$. Then the extremal ray $R \subset NE(\mathcal{Y}_i/\mathcal{C})$ inducing $\mathcal{Y}_i \dashrightarrow \mathcal{Y}_{i+1}$ is also $(K_{\mathcal{Y}_i} + \Delta_i)$ -negative, for every $i < m$. Moreover, $\Delta_m = (\mathcal{Y}_m)_{s,\text{red}}$ because all the components of B are contracted on \mathcal{Y}_m . Thus (4.9) is also an MMP-sequence for (\mathcal{Y}, Δ) .

We define the skeleton $\text{Sk}(\mathcal{Y}_i) \subset X^{\text{an}}$ as in [NX13, 3.1.2], and we again denote by $\text{Sk}(\mathcal{Y}_i)^{\leq w}$ the subcomplex of $\text{Sk}(\mathcal{Y}_i)$ generated by the vertices where the value of wt_ω is at most w . Then $\text{Sk}(\mathcal{Y}_i)^{\leq w}$ is canonically homeomorphic to the dual complex of A_i . Let $f_i : \mathcal{Y}_i \dashrightarrow \mathcal{Y}_{i+1}$ be one of the maps in the sequence produced by the MMP, corresponding to an extremal ray R . We claim that either f_i does not contract any log canonical center of A_i or $R \cdot E > 0$ for some prime component E of A_i . In the former case, $\text{Sk}(\mathcal{Y}_i)^{\leq w} = \text{Sk}(\mathcal{Y}_{i+1})^{\leq w}$. In the latter case, it follows from [dFKX12, Thm. 19] that $\text{Sk}(\mathcal{Y}_{i+1})^{\leq w}$ is an elementary collapse of $\text{Sk}(\mathcal{Y}_i)^{\leq w}$. Thus it suffices to prove our claim. The following argument is a variant of the proof of Lemma 21 in [dFKX12].

By the definition of the weight function, the divisor $K_{\mathcal{Y}_i/\mathcal{C}} + (\mathcal{Y}_i)_{s,\text{red}}$ is \mathbb{Q} -linearly equivalent over \mathcal{C} to a \mathbb{Q} -divisor $D_1 - D_2$ such that D_1 and D_2 are effective, D_1 is supported on B_i and D_2 is supported on A_i . Now we can write

$$(4.10) \quad 0 > R \cdot (K_{\mathcal{Y}_i/\mathcal{C}} + \Delta_i) = R \cdot (D_1 - \varepsilon B_i - D_2).$$

Assume that f_i contracts a log canonical center W of the divisor A_i . By choosing $\varepsilon > 0$ sufficiently small, we can assume that the divisor $D_1 - \varepsilon B_i \geq 0$ is supported on B_i and therefore does not contain W . Thus for every curve Y in $(\mathcal{Y}_i)_s$ through a general point of W , we have $Y \cdot (D_1 - \varepsilon B_i) \geq 0$. It follows that $R \cdot (D_1 - \varepsilon B_i) \geq 0$ which implies that $R \cdot D_2 > 0$ because of (4.10). This concludes the proof. \square

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